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Free Prounipotent Groups

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A prounipotent group (over $\mathbb{C}$) is a pro-affine algebraic group which is an inverse limit of unipotent complex algebraic groups. These groups were studied in [7], where the existence of free prounipotent groups on a set was established. In this paper we analyze the structure, lower central series, automorphism group, and some abstract subgroups, of a free prounipotent group on d generators.

When we introduced free prounipotent groups in [7], they were constructed in terms of their universal mapping property. In seeking to understand their internal structure, which is our goal in this paper, we have been guided by the theory of discrete free groups. In particular, we have obtained the analogue of Magnus' embedding theorem, which embeds the discrete free group on generators $g_1,..., g_d$ in the algebra $\mathbb{Z}\langle\langle t_1,..., t_d\rangle\rangle$ of formal power series in the non-commuting indeterminants $t_1,..., t_d$ such that g_p is sent to $1 + t_p$ [8, Theorem 5.6, p. 308]. We show here that the free prounipotent group G on $g_1,..., g_d$ is similarly embedded in $\mathbb{C}\langle\langle t_1,..., t_d\rangle\rangle$.

This embedding theorem appears quite naturally in our setting, for reasons we now explain: let E denote the coordinate ring of G. E is a two-sided G-module, so right G-translations on E are endomorphisms of E as a left G-module. Thus, we have G embedded in $\mathrm{End}_G(E)$ as right translations. (So far, G could be any proalgebraic group.) What we prove here, for G free prounipotent on $g_1,..., g_d$, is that $\mathrm{End}_G(E) = \mathbb{C}\langle\langle t_1,..., t_d\rangle\rangle$ where $1 + t_p$ is right translation by g_p. We call this result the endomorphism structure theorem. (The algebra $\mathrm{End}_G(E)$ is important even when G is not free since G-

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modules are also $\mathrm{End}_G(E)$-modules. In fact, $\mathrm{End}_G(E)$ is much like a group algebra for G. In a subsequent paper we use this algebra for further applications to cohomology and combinatorial group theory of prounipotent groups.)

The endomorphism structure theorem is established in the first section of this paper below. In the second section, we consider the lower central series of the free prounipotent group G: $\{C^iG \mid i = 1, 2,...\}$. (Initially, this is taken in the sense of pro-affine algebraic groups, although we subsequently show that in fact it agrees with the abstract lower central series.) Using the endomorphism structure theorem we prove, similarly to the discrete case, that the complex Lie algebra $\mathrm{gr}(G) = \bigoplus \{C^iG/C^{i+1}G \mid i \geqslant 1\}$ is free on the d generators $g_p C^2G$.

With this analysis of the lower central series of G available, we can analyze $\mathrm{Aut}(G)$ in much the same way that the automorphism group of a discrete free group is analyzed in [1], which we do in our third section. The fourth section of the paper deals with some abstract subgroups of G. In the final section, we use properties of one of these subgroups (a d-fold free product of copies of $\mathbb{Q}$) to give a new proof of the Malcev embeddings of finitely generated torsion free nilpotent groups into divisible groups and algebraic groups.

We have stayed over $\mathbb{C}$ in this paper for concreteness. The results pass without change to arbitrary algebraically closed characteristic zero fields.

This paper relies on our previous work [7] although we have departed somewhat from [7] in our notations here. Our notational conventions are as follows: let H be a prounipotent group. $\mathbb{C}[H]$ denotes the *coordinate ring* of H; if $f \in \mathbb{C}[H]$ an $h \in H$, then $f \cdot h$ and $h \cdot f$ are given by $(f \cdot h)(x) = f(hx)$ and $(h \cdot f)(x) = f(xh)$. $\mathbb{C}[H]$ is an H-module under the *left action* $h(g) = h \cdot f$ and the *right action* $h(f) = f \cdot h^{-1}$. G_a denotes the additive (algebraic) group $\mathbb{C}$ and an *additive character* of H is a homomorphism (of prounipotent groups) $H \to G_a$. $X^+(H)$ is the complex vector group of additive characters. If Γ is any group, $C^1\Gamma = \Gamma$ and $C^{i+1}\Gamma = (\Gamma, C^i\Gamma)$, where the commutator (x, y) is $x^{-1}y^{-1}xy$. For H, $C^1H = H$ and $C^{i+1}H$ is the Zariski-closure of (H, C^iH). A *central filtration* of Γ is a set of subgroups $\{\Gamma_i\}$ with $(\Gamma_i, \Gamma) \subseteq \Gamma_{i+j}$ and the *associated graded* group (actually Lie algebra) is $\mathrm{gr}(\Gamma) = \bigoplus \{\Gamma_i/\Gamma_{i+1} \mid i \geqslant 1\}$. When no central filtration for H (or Γ) is specified, $\mathrm{gr}(H)$ (or $\mathrm{gr}(\Gamma)$) is defined from $\{C^iH\}$ (or $C^i\Gamma$). If L is a Lie algebra, $\mathbb{C}^1L = L$ and $\mathbb{C}^{i+1}L = [L, L^i]$. We use "$\langle\langle \cdots \rangle\rangle$" for non-commutative formal power series. Homomorphisms between prounipotent groups are assumed algebraic, and unipotent groups are finite-dimensional prounipotent.

1. Coordinate Ring Endomorphisms and the Embedding Theorem

In this section, we show that the free prounipotent group G on d generators can be embedded in the multiplicative group of complex power series, of constant term 1, in d non-commuting variables (Embedding Theorem). We proceed by showing that the G-module endomorphism ring of the coordinate ring of G is the full power series algebra (Endomorphism Structure Theorem). This approach shows how automorphisms of G extend naturally to automorphisms of the power series group.

Notation 1.1. G is a free unipotent group on the generators $g_1,..., g_d$. E is the coordinate ring of G regarded as a left G-module. For $p = 1,..., d$, $X_p \in \mathrm{End}_G(E)$ is defined by $X_p(f) = f \cdot g_p^{-1} - f$. We define subspaces E_i of E inductively as follows: $E_{-1} = 0$, and $E_i = \{f \in E \mid X_p(F) \in E_{i-1}$ for $p = 1,..., d\}$.

The subspaces introduced in (1.1) have a number of elementary properties which we now record:

Lemma 1.2. *In notation* 1.1

(1) *Each E_i is a G-submodule of E.*

(2) $\bigcup \{E_i \mid i \geqslant -1\} = E$ *and* $E_i \subseteq E_{i+1}$.

(3) $E_i = \{f \in E \mid mf = 0$ *for all monomials m in $X_1,..., X_d$ of degree* $i + 1\}$.

Proof. (1) We proceed by induction on i, the case $i = -1$ being trivial. Suppose (1) holds for $i = 1$, and let $f \in E_i$. For each p, and each $h \in G$, $X_p(h \cdot f) = h \cdot (f \cdot g_p^{-1} - f) \in h \cdot E_{i-1} = E_{i-1}$, so $h \cdot f \in E_i$.

(2) Let $f \in E$ and let M be the right G-submodule of E generated by f. Define an ascending sequence $s^i(M)$ of right G-submodules of M by setting $s^{-1}(M) = 0$ and $s^{i+1}(M)/s^i(M) = (M/s^i(M))^G$. If $f \in s^i(M)$, then for each p we have $X_p(f) \in s^{i-1}(M)$, so since $s^{-1}(M) \subseteq E_{-1}$, it follows by induction that $s^i(M) \subseteq E_i$. Since M is finite-dimensional and G is prounipotent, we have $M = s^i(M)$ for some i, so $f \in E_i$. Thus the first assertion of (2) is obtained. It is clear that E_0 is the constants, so $E_i \subseteq E_{i+1}$ for $i = -1$. Assume that $E_i \subseteq E_{i+1}$ for some i, and let $f \in E_{i+1}$. Then $X_p f \in E_{i+1}$ for each p, so $f \in E_{i+2}$. Thus $E_{i+1} \subseteq E_{i+2}$ and we obtain the second assertion of (2) by induction.

(3) Use induction on i, the case $i = -1$ being clear. Suppose it holds for the case $i - 1$, and let f in E be annihilated by all monomials of degree $i + 1$. Then for each p, $X_p f$ is annihilated by all monomials of degree i, so $X_p f \in E_{i-1}$ by the induction assumption. Then $f \in E_i$, so the left-hand side of (3) contains the right. A similar induction establishes the reverse containment.

Next, we count the dimensions of the E_i and examine the quotients E_{i+1}/E_i.

LEMMA 1.3. *In Notation* 1.1, *we define* $\Phi_1\colon E \to E^{(d)}$ *by* $\Phi_1(f) = (X_1 f, \dots, X_d f)$. *Then*:

(1) Φ_1 is surjective with kernel E_0.

(2) $\dim(E_i) = 1 + d + d^2 + \cdots + d^i$ *for* $i \geqslant 0$.

(3) *Regarding* E/E_i *as a left* G-*module, we have* $(E/E_i)^G = E_{i+1}/E_i$ *for* $i \geqslant -1$.

Proof. We begin by computing E_1 : f is in E_1, if and only if $X_p f - f$ is constant for $p = 1, \dots, d$ so $f \cdot g - f = a(g)$ where $a \in E$. For all g, h in G we then have $f(gh) = f(h) + a(g)$ so taking $h = e$ we find $f(g) - f(e) = a(g)$, so $f(gh) = f(g) + f(h) - f(e)$. Set $\chi = f - f(e)$. Then χ is an additive character of G. Conversely, if χ is any additive character of G and a is a constant, then $\chi + a$ is in E_1. Thus $E_1 = E_0 \oplus X^+(G)$, where $X^+(G)$ is the (d-dimensional) group of additive characters of G. We note that if $f \in E_1$ and $g \in G$, then $gf - f \in E_0$, and since the above arguments are right–left symmetric, if $f \in E$ and $g \cdot f - f \in E_0$ for all g in G, then $f \in E_1$. We have verified assertion (2) for $i = 0, 1$ and (3) for $i = 0$.

It follows from (1.2)(3) that Φ_1 has kernel E_0. E is the injective hull of E_0, and since G has cohomological dimension 1 [8, Theorem 2.9], we have E/E_0 injective, so $E/E_0 = E^{(m)}$ for $m = \dim((E/D_0)^G)$. By the above, we know $m = d$. It follows that Φ_1 is surjective. From (1.2)(3), it follows that $\Phi_1(E_{i+1}) = E_i^{(d)}$ for $i \geqslant 0$, and by (1.2)(2), $E_0 \subseteq E_{i+1}$ for $i \geqslant 0$. The dimension formula (2) now follows by induction on i. For $i \geqslant 0$, the inclusion $E_0 \subseteq E_i$ implies that Φ_1 induces an isomorphism $E/E_{i+1} \to (E/E_i)^{(d)}$ of left G-modules, and now (3) follows by induction on i from the case $i = 0$ considered in the first paragraph.

We can now determine the quotients E/E_i.

LEMMA 1.4. *In Notation* 1.1, *we order* $X_1, \dots, X_d$ *so that* $X_p < X_{p+1}$. *This induces the lexocographic order on the monomials* M_i *of degree* i *in* $X_1, \dots, X_d$: $M_i = \{m_1, \dots, m_{d^i}\}$. *Define* $\Phi_i\colon E \to E^{(d^i)}$ *by* $\Phi_i(f) = (m_1 f, \dots, m_{d^i} f)$. *Then* Φ_i *is surjective with kernel* E_{i-1}, *and in particular, there are elements* $f_1, \dots, f_{d^i}$ *of* E_i *such that* $m_r f_s = \delta_{r,s}$ *for all* $1 \leqslant r, s \leqslant d^i$.

Proof. By (1.3)(3) and (1.3)(2), $(E/E_{i-1})^G$ has dimension d^i, so since E/E_{i-1} is injective, $E/E_{i-1} = E^{(d^i)}$, and Φ_i has kernel E_{i-1} by (1.2)(3), it follows that Φ_i is surjective. Choose $f_1, \dots, f_{d^i}$ in E such that $\Phi_i(f_1), \dots, \Phi_i(f_{d^i})$ are the standard basis (in order) of $(E/E_{i-1})^G = E_0^{(d^i)}$. By (1.3)(3) we have each f_s in E_i, and by choice of f_s we have $m_r f_s = \delta_{r,s}$ for all r, s.

The data obtained in the three lemmas above allow us to determine the structure of the endomorphism ring $\mathrm{End}_G(E)$:

THEOREM 1.5 (Endomorphism Structure Theorem). *There is a* $\mathbb{C}$-*algebra isomorphism* $F\colon \mathbb{C}\langle\langle t_1,..., t_d\rangle\rangle \to \mathrm{End}_G(E)$ *which sends* t_p *to* X_p *of* (1.1).

Proof. An induction argument and (1.3)(3) shows that if $T \in \mathrm{End}_G(E)$, then $T(E_i) \subseteq E_i$ for all i. Since E is injective, $\Psi_i\colon \mathrm{End}_G(E) \to \mathrm{Hom}_G(E_i, E)$ is surjective, and the preceding observation shows that the image of Ψ_i is $\mathrm{End}_G(E_i)$. By (1.2)(2) we have $\mathrm{End}_G(E) = \mathrm{Hom}_G(\text{dir lim } E_i, E) = \text{proj lim}$ $\mathrm{Hom}_G(E_i, E)$, so the Ψ_i induce an isomorphism $\Psi\colon \mathrm{End}_G(E) \to \text{proj lim}$ $\mathrm{End}_G(E_i)$.

Next we define F: if $f \in E$ and $a \in A = \mathbb{C}\langle\langle t_1,..., t_d\rangle\rangle$, then $f \in E_i$ for some i (1.2)(2) so $a(X_1,..., X_d)(f)$ makes sense: terms of $a(X_1,..., X_d)$ of degree exceeding i vanish on f by (1.2)(3). Thus $F(a) = a(X_1,..., X_d)$ is well-defined and is clearly an algebra homomorphism. It further follows from (1.2)(3) that $\Psi_i F$ vanishes on all monomials of degree exceeding i. Let I_i be the ideal of A of all formal series whose lowest-degree homogeneous term has degree exceeding i. Then $\Psi_i F$ vanishes on I_i, and A/I_i has a basis given by the images of the monomials of degree i or less. Let a in A be in the linear span of these monomials and suppose $\Psi_i F(a) = 0$. Let m be a monomial appearing in a with non-zero coefficient and of minimal degree j. By (1.4), there is $f \in E_j$ such tht $F(m)f = 1$ and $F(m')f = 0$ for every other monomial of degree j. By (1.2)(3), and minimality of j, we then have $0 = \Psi_i F(a)$ equal to the coefficient of m in a, contrary to choice of m. It follows that $a = 0$, so $\Psi_i F$ induces an injection $A/I_i \to \mathrm{End}_G(E_i)$. By (1.3)(2), we have $\dim(A/E_i) = \dim(E_i)$. Now $\mathrm{End}_G(E_i) = \mathrm{Hom}_G(E_i, E)$, and E is the G-module induced from the trivial $\mathbb{C}$-module [5, Proposition 1.4, p. 3], so $\mathrm{Hom}_G(E_i, E) = \mathrm{Hom}_{\mathbb{C}}(E_i, \mathbb{C})$ and $\dim(\mathrm{End}_G(E_i)) = \dim(E_i)$. Thus $\Psi_i F$ induces an isomorphism $A/I_i \to \mathrm{End}_G(E_i)$. Since $A = \text{proj lim}(A/I_i)$ and $\mathrm{End}_G(E) = \text{proj lim } \mathrm{End}_G(E_i)$, we see that ΨF, and hence F, is an isomorphism.

The group G is embedded in $\mathrm{End}_G(E)$ via the right translation action, and under this embedding the generators g_p map to the endomorphisms $1 + X_p$. As a consequence of (1.5), we then have G embedded in the multiplicative group of formal power series of constant term 1. We record this corollary:

THEOREM 1.6 (Embedding Theorem). *Let G be the free prounipotent group on $g_1,..., g_d$. Then there is an embedding θ of G in the multiplicative group of formal power series, in the d non-commuting indeterminates $t_1,..., t_d$, of constant term* 1 *which sends g_p to* $1 + t_p$.

The analogue of (1.6) for discrete free groups is [8, Theorem 5.6, p. 308].

We need to comment further on the range of θ in (1.6). It will be convenient to fix some additional notation for future use.

Notation 1.7. In Notation 1.1, $J_{i+1} = \{f \in E \mid f(E_i) = 0\}$, and $1 + J_i$ denotes the multiplicative subgroup of $\mathrm{End}_G(E)$ consisting of all f such that $f - 1 \in J_i$.

In the proof of (1.5) we actually identified J_i: it is the formal power series in $X_1, \ldots, X_p$ whose lowest-degree homogeneous term has degree at least i. This means that if $f, g \in J_i$, then $fg \in J_{i+1}$. It then follows that the function $1 + J_i \to J_i/J_{i+1}$ which sends f to $(f-1) + J_{i+1}$ in a surjective group homomorphism with kernel $1 + J_{i+1}$. (Here, we regard J_i/J_{i+1} as a vector groups.) An easy induction now shows that $(1 + J_1)/(1 + J_{i+1})$ is a unipotent algebraic group of dimension $d + d^2 + \cdots + d^i$. Since $\bigcap \{i \mid i + J_i\} = 1$, we see that $1 + J_1$ is prounipotent and it is clear that $\theta \colon G \to 1 + J_1$ is a morphism. Finally, let us note that $1 + J_i$ is Zariski-closed in $1 + J_1$. For use in Section 2 below, we record the following commutator formulae for the groups $1 + J_i$:

LEMMA 1.9. *In Notation* 1.7, *let* $x \in J_r$ *and* $y \in J_s$. *Then* $(1 + x, 1 + y) = 1 + z$ *where* $z \in J_{r+s}$ *and* $z \equiv xy - yx (\mathrm{mod}\, J_{r+s+1})$.

Proof. This is an elementary computation; see [9, p. La. 4.11].

Our proof of (1.6) via (1.5) also displays a functorial property: if σ is an automorphism of G, σ defines an automorphism (which we shall call σ) of E which sends f to $f \cdot \sigma$. Then for $g \in G$, $\sigma(gf) = \sigma^{-1}(g)\,\sigma(f)$, and it follows that sending T to $\sigma^{-1} T \sigma$ defines an automorphism of $\mathrm{End}_G(E)$, such that $\sigma^{-1} X_p \sigma(f) = f \cdot \sigma(g_p^{-1}) - f$. There is thus an automorphism s of $\mathbb{C}\langle\langle t_1, \ldots, t_d \rangle\rangle$ such that $F(s(tp)) = \sigma^{-1} X_p \sigma$. Of course, s is also an automorphism of $1 + J_1$, and we have $s\theta(g_p) = \theta(\sigma(g_p))$ for $p = 1, \ldots, d$. We have thus established the following corollary to (1.6).

COROLLARY 1.10. *If σ is an automorphism of the free prounipotent group G on $g_1, \ldots, g_d$, then there is an automorphism s of the algebra $\mathbb{C}\langle\langle r_1, \ldots, t_d \rangle\rangle$ such that $s\theta(x) = \theta(\sigma(x))$ for all x in G, where θ is as in* (1.6).

Note that (1.10) gives a generalization to prounipotent groups of § 11 of [1].

2. THE LOWER CENTRAL SERIES

In this section we calculate the lower central series of the free unipotent group G on d generators, in the following sense: we show that the associated graded Lie algebra of the filtration of G by the closed lower central series is the free complex Lie algebra on d generators. Our method is to proceed, as in the discrete free group case, via the Embedding Theorem. This parallel approach allows us to keep track of the lower central series of the discrete free group F generated by the generators of G: we find that $C^iF = C^iG \cap F$ and C^iF is Zariski-dense in C^iG. Once we have determined the closed lower central series of G, we use that determination to show that it agrees with the abstract central series of G.

It follows from (1.9) that $\theta^{-1}(1+J_r)$ contains C^rG, where θ is as in (1.6) and G is as in (1.1), so that $C^{r+1}G$ acts trivially on E_r under the right action. We will show that $G/C^{r+1}G$ acts faitfully on E_r, so that $\theta^{-1}(1+J_r) = C^rG$. The following notation is adopted:

Notation 2.1. In Notation 1.1, $G(r) = G/C^{r+1}G$, $D_p = \sum_{i=1}^{\infty} (-1)^{i-1}\, iX_p^i$ for $p = 1,..., d$. L is the Lie subalgebra of $\mathrm{End}_G(E)$ generated by $D_1,..., D_p$ and $L(r) = L/\mathscr{C}^{r+1}L$.

We first show that L is a free $\mathbb{C}$-Lie algebra on $D_1,..., D_p$.

LEMMA 2.2. *In Notation* 2.1:

(1) *The associative subalgebra of* $\mathrm{End}_G(E)$ *generated by* $D_1,..., D_p$ *is a free associative algebra on* $D_1,..., D_p$.

(2) *L is the free $\mathbb{C}$-Lie algebra on* $D_1,..., D_p$.

Proof. Let $m = m(D_1,..., D_p)$ be a monomial in $D_1,..., D_p$ of degree i. It follows from (1.2)(3) that for $f \in E_i$, $mf = m(X_1,..., X_p)f$. It follows then from (1.4) that the monomials of degree i in $D_1,..., D_p$ are linearly independent, and now (2.2)(1) follows. Then (2.2)(2) follows from (2.2)(1) by [4, Theorem 1, p. 136].

COROLLARY 2.3. *In Notation* 2.1, *$L(r)$ is isomorphic to* $\mathrm{Lie}(G(r))$.

Proof. $G(r)$ is an unipotent group, $L(r)$ is a finite-dimensional Lie algebra, and the isomorphism follows from their respective universal properties.

The isomorphism of (2.3) can be made explicit: each D_p is a derivation of E satisfying $D_p(x \cdot f) = x \cdot D_p(f)$ for $x \in G$ and $f \in E$, and $\exp(D_p) = 1 + X_p$ so $\exp(D_p)(f) = f \cdot g_p^{-1}$ for $f \in E$. We can identify $\mathbb{C}[G(r)]$ with the $C^{r+1}G$-invariants of E, which we denote by $E(r)$, and then L restricts to a Lie algebra $\bar{L}$ of derivtions of $E(r)$ such that each D in $\bar{L}$ satisfies $D(x \cdot f) = x \cdot D(f)$ for $x \in G(r)$ and $f \in E(r)$. It follows that $\bar{L} \subseteq \mathrm{Lie}(G(r))$, for example by [6, p. 65]. (The reader is warned that we are using "left and right actions" in the opposite sense of that of [6].) Let $\bar{D}_p = D_p \mid E(r)$ and $\bar{g}_p$ be the image of g_p in $G(r)$. Then $\exp(\bar{D}_p)(f) = f \cdot \bar{g}_p^{-1}$ for $f \in E(r)$, so since

$\bar{g}_1,\ldots,\bar{g}_p$ generate $G(r)$, $\bar{D}_1,\ldots,\bar{D}_p$ generate Lie$(G(r))$. Thus $\bar{L} = \mathrm{Lie}(G(r))$, and $L \to \bar{L}$ is onto with kernel $\mathbb{C}^{r+1}L$ by (2.3).

We can now see that E_r is a faithful $G(r)$ module, under the right action.

LEMMA 2.4. *In Notation* 2.1, *the restriction* $L \to \mathrm{End}_G(E_r)$ *has kernel* $\mathbb{C}^{r+1}L$.

Proof. Since $\mathscr{C}^{r+1}L \subseteq J_{r+1}$ (Notation 1.7) and the restriction $\mathrm{End}_G(E) \to \mathrm{End}_G(E_r)$ has kernel J_{r+1}, it will suffice to show that $J_{r+1} \cap L \subseteq \mathbb{C}^{r+1}L$. Choose a Hall basis H for L containing $D_1,\ldots, D_p$ [4, Definition 3, p. 136], let U be the associative subalgebra of $\mathrm{End}_G(E)$ generated by $D_1,\ldots, D_p$ and let I be the ideal of U generated by $D_1,\ldots, D_p$. Since $D_p = X_p(\sum_{i=1}^{\infty} (-1)^{i-1}\, iX_p^{i-1})$, the monomials in D_p of degree i generate J_i as an ideal of $\mathrm{End}_G(E)$, and it follows that $J_i \cap U = I^i$. Let $H_0 = \{x \in H \mid \mathrm{length}(X) \leqslant r\}$, and let $Y = \sum a_X X$ be a linear combination of elements of H_0. If $Y \in J_{r+1} \cap L$, then $Y \in I^{r+1}$, so Y is a linear combination of associative monomials in $D_1,\ldots, D_p$ of degree exceeding $r+1$. But each X in H_0 is a combination of associative monomials in $D_1,\ldots, D_p$ of degree less than r, so by (2.2)(1) $Y = 0$ and hence each $a_X = 0$. If $Z \in J_{r+1} \cap L$, $Z = Y_0 + Y_1$, where Y_0 is a linear combination of elements of H_0 and Y_1 is a linear combination of elements $H_1 = H - H_0$. Since $H_1 \subseteq \mathbb{C}^{r+1}L \subseteq J_{r+1}$, we have $Y_0 \in J_{r+1} \cap L$ so $Y_0 = 0$ by the above and $Z = Y_1 \in \mathscr{C}^{r+1}L$.

COROLLARY 2.5. *In Notation* 2.1, E_r *is a faithful* $G(r)$*-module, under the right action.*

Proof. Let ρ: $G(r) \to \mathrm{End}_{\mathbb{C}}(E_r)$ be the corresponding representation. From (2.3) and the remarks following, we know that the differential of ρ $d\rho$: $\mathrm{Lie}(G(r)) \to \mathrm{End}_{\mathbb{C}}(E_r)$ is given as follows: $\mathrm{Lie}(G(r)) = L(r) = L \mid E_r$ and for $D \in L$, $d\rho(D \mid E_r) = D \mid E_r$. By (2.4), $d\rho$ is an injection on $L(r)$, so (2.5) is obtained.

We have now identified the lower central series of G:

PROPOSITION 2.6. *In Notation* 1.7, $\theta^{-1}(1 + J_r) = C^rG$, *where* θ *is the embedding of* (1.6).

Proof. As previously noted, (1.9) implies that $\theta^{-1}(1 + J_r)$ contains C^rG. Since the kernel of $1 + J_1 \to \mathrm{End}_G(E_{r-1})$ is $1 + J_r$, and E_{r-1} is a faithful $G(r-1)$-module by (2.5), $\theta^{-1}(1 + J_r) = C^rG$.

We can now compute the associated grated Lie algebra [3, p. 146] of the filtration $\{C^rG\}$ and its relation to the associated graded Lie algebra of the discrete free group.

THEOREM 2.7. *Let* G *be the free prounipotent group with generators*

$g_1, \ldots, g_d$ and let F be the discrete subgroup of G generated by $g_1, \ldots, g_d$. Then:

(1) *$C^rG \cap F = C^rF$ and G^rF is Zariski-dense in C^rG for $r = 1, 2, \ldots$.*

(2) *$\mathrm{gr}(G) = \bigoplus \{C^rG/C^{r+1}G \mid r \geqslant 1\}$ is the free $\mathbb{C}$-Lie algebra on $\{g_pC^2G \mid p = 1, \ldots, d\}$.*

Proof. In Notation 1.7, let $\theta\colon G \to 1 + J_1$ be the embedding of (1.6). By (1.9), $\{1 + J_r\}$ is a filtration of $1 + J_1$ in the sense of [4, Definition 2, p. 145]. For $r \geqslant 1$, let J^r be the span of the monomials of degree r. $J^r \to J_r/J_{r+1}$ is an isomorphism of vector spaces, and we saw in the remarks preceding (1.9) that $1 + J_r/1 + J_{r+1} \to J_r/J_{r+1}$ induced from the map $z \to z - 1$ is a group isomorphism. Thus, $\mathrm{gr}(1 + J_1) = \bigoplus (1 + J_r/1 + J_{r+1})$ is isomorphic as abelian group to $\bigoplus \{J^r \mid r \geqslant 1\} = I \subseteq J_1$. Now $\mathrm{gr}(1 + J_1)$ is a Lie algebra [3, Proposition 1, p. 145] and it follows from (1.9), $\mathrm{gr}(1 + J_1)$ is isomorphic to I as Lie algebra, where for $x, y \in I$ we have $[x, y] = xy - yx$. In particular, we note that I is a complex Lie algebra, and the complex sub-Lie algebra generated by $t_1, \ldots, t_d$ is free. (Note also that t_i is the image of $(1 + t_i)(1 + J_2)$ in our identifications.) By (2.6), $\theta^{-1}(1 + J_r) = C^rG$, and by [4, Theorem 2, p. 152], $C^rF = \theta^{-1}((1 + J_r) \cap \theta(F))$. We have, thus, Lie algebra injections $\mathrm{gr}(F) \to \mathrm{gr}(G) \to \mathrm{gr}(1 + J_1) = I$, where we use the filtrations $\{C^rF\}$ and $\{C^rG\}$ on F and G, respectively. We have now seen that $C^rG \cap F = C^rF$, also. The map $\mathrm{gr}(G) \to I$ is an injection of complex vector spaces, so that $\mathrm{gr}(G)$ is a complex Lie algebra. By [4, Theorem 3, p. 152], $\mathrm{gr}(F)$ is a free $\mathbb{Z}$-Lie algebra, and $C^rF/C^{r+1}F$ is free abelian of rank $l_d(r)$ [9, Theorem 4.2, p. La. 4.4]. Using Notation 2.10, we know from (2.3) that $\mathrm{Lie}(C^rG/C^{r+1}G) = \mathscr{C}^rL/\mathscr{C}^{r+1}L$, which is of dimension $l_d(r)$ [9, Theorem 4.2, p. La. 4.4]. Thus $(C^rF/C^{r+1}F) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^rG/\mathbb{C}^{r+1}G$, so $\mathrm{gr}(G) = \mathrm{gr}(F) \otimes \mathbb{C}$ is a free complex Lie algebra on $\{g_p\mathbb{C}^2G \mid p = 1, \ldots, d\}$, so (2.7)(2) holds. To complete the proof, we need that C^rF is Zariski-dense in C^rG. We proceed by induction, the case $r = 1$ being clear. Assume it holds for $r - 1$, and let H be the Zariski-closure of C^rF in G. We know that $H \subseteq C^rG$ and that H is normal in G. Then $\bar{F} = C^{r-1}F/C^rF$ is Zariski-dense in $\bar{G} = C^{r-1}G/H$, and $\bar{F}$ is abelian, so $\bar{G}$ is a complex vector space of dimension $l \leqslant l_d(r - 1)$. Since $\bar{G}$ maps surjectively to $C^{r-1}G/C^rG$, we also know that $l \geqslant l_d(r - 1)$. It follows that $l = l_d(r - 1)$ and that $H = C^rG$ so C^rF is Zariski-dense in C^rG.

For use below, we note the following formula used in the proof of (2.7):

Corollary 2.8. *In Notation* 1.1, *$C^rG/C^{r+1}G$ has dimension* $l_d(r) = r^{-1} \sum_{m \mid r} \mu(m)\, d^{m/r}$ [9, p. La. 4.4].

We can regard (2.7) as yielding a description of the lower central series of G, in the sense of proalgebraic groups. We shall now see that this is actually the same as the lower central series of G as an abstract group. The method, basically, is an elementary adaption of [3, Theorem 2.1, p. 221]. We introduce some provisional notation for this abstract central series:

Notation 2.9. In Notation 1.1, let $G_1 = G$ and $G_{i+1} = (G, G_i)$ for $i \geqslant 1$. We note that $G_i \subseteq C^i G$ and we have equality for $i = 1$. To apply the proof of [3, Theorem 2.1, p. 221], we require the following consequence of (2.7).

LEMMA 2.10. *In Notation* 1.1, *if* $a \in C^k G$ *for* $k \geqslant 2$, *there are* $u_1, \ldots, u_d \in C^{k-1} G$ *such that* $a \equiv (u_1, g_1) \cdots (u_d, g_d) \pmod{C^{k+1} G}$.

Proof. By (2.7)(2), $C^k G / C^{k+1} G$ is the degree k homogeneous piece of the free $\mathbb{C}$-Lie algebra on $\{g_p C^2 G\}$, and it is immediate that every element of this piece has the desired form.

THEOREM 2.11. *In Notation* 1.1, $C^i G = (G, (G, (\ldots, G)) \ldots)$ (*i terms*).

Proof. Following [3, Theorem 2.1, p. 221], choose $a \in C^i G$. We will find elements $B^{(p)}$ in $C^{i-1} G$ such that $a = (B^{(1)}, g_1) \cdots (B^{(d)}, g_d)$. By (2.9), there are elements $b_1^{(p)} \in C^{i-1} G$ such that $a \equiv \prod (b_1^{(p)}, g_p) \pmod{C^{i+1} G}$. Suppose we have found elements $b_1^{(p)}, \ldots, b_k^{(p)}$ such that $b_{l+1}^{(p)} \equiv b_l^{(p)} \pmod{C^{i+l-1} G}$ for $1 \leqslant l \leqslant k-1$, and $a \equiv \prod (b_k^{(p)}, g_p) \pmod{C^{i+k} G}$. Thus $a = \prod (b_k^{(p)}, g_p)\, s$ for $s \in C^{i+1} G$. By (2.9), $s \equiv (u_1, g_1) \cdots (u_p, g_p) \pmod{C^{i+k+1}}$ with $u_p \in C^{i+k-1} G$. Set $b_{k+1}^{(p)} = b_k^{(p)} u_p$, so $b_{k+1}^{(p)} \equiv b_k^{(p)} \pmod{C^{i+k-1} G}$. Then $\prod (b_{k+1}^{(p)}, g_p) = \prod (b_k^{(p)} u_p, g_p) = \prod ((b_k^{(p)}, g_p)^{u_p} (u_p, g_p))$ by [3, Lemma 2.3, p. 221]. Since $u_p \in C^{i+k-1}$ and $(b_k^{(p)}, g_p) \in C^2 G$, we have $(u_p, (b_k^{(p)}, g_p)) \in C^{i+k+1} G$ and $(u_p, g_p) \in C^{i+k} G$ so modulo $C^{i+k+1} G$ the first commutator is trivial and the second central, and modulo $C^{i+k+1} G$ we have $\prod (b_{k+1}^{(p)}, g_p) \equiv (\prod (b_k^{(p)}, g_p))(\prod (u_p, g_p)) \equiv \prod (b_k^{(p)}, g_p)\, s = a$. Thus inductively we can find sequences of elements $b_k^{(p)}$ of $C^{i-1} G$ such that $b_{k+1}^{(p)} \equiv b_l^{(p)} \pmod{C^{i+k-1} G}$ and $a \equiv \prod (b_k^{(p)}, g_p) \pmod{C^{i+k} G}$. Since $C^{i-1} G = \operatorname{proj\,lim}(C^{i-1} G / C^{i+k-1} G)$, there are elements $B^{(p)}$ of G such that $B^{(p)} \equiv b_k^{(p)} \pmod{C^{i+k-1} G}$ for all k. It then follows that $(B^{(p)}, g_p) \equiv (b_k^{(p)}, g_p) \pmod{C^{i+k} G}$ so $a = \prod (B^{(p)}, g_p)$. This proves that $C^i G \subseteq (G, C^{i-1} G)$, and hence that $C^i G = (G, C^{i-1} G)$. Since $C^1 G = G_1$, induction shows that $C^i G = G_i$.

The universal property of free prounipotent groups now show that (2.10) applies to all finitely generated prounipotent groups.

COROLLARY 2.12. *Let H be a finitely generated prounipotent group. Then for* $i \geqslant 1$, $C^i H = (H, (H, (\ldots, H)) \ldots)$ (*i terms*).

Proof. Assume H is d generated and choose a surjective homomorphism $f\colon G \to H$. Then $f(C^i G) = G^i H$, and $f(G_i) = (H, (H, (\ldots, H)), \ldots)$ (i terms) so the result follows from (2.11).

3. Automorphisms of Free Prounipotent Groups

In this section we consider the group $A = \mathrm{Aut}(G)$ of automorphisms of the free prounipotent group G on $d \geqslant 2$ generators. A has a chain of normal subgroups $\{K_i \mid i = 1, 2, \ldots\}$ where K_i is the group of elements of A acting trivially on $G/C^{i+1}G$. Since G/C^2G is a d-dimensional vector group, we have $A/K_1 = GL_d(\mathbb{C})$. We show that K_1 is prounipotent, so K_1 is the prounipotent radical of A and $A = K_1 \cdot GL_d(\mathbb{C})$ (semi-direct product). $\{K_i\}$ is a central filtration of K_1; we compute the quotients K_i/K_{i+1} and their dimensions. From this we conclude that K_1 always contains outer automorphisms (even for $d = 2$). We also identify the associated graded Lie algebra $\bigoplus \{K_i/K_{i+1} \mid i \geqslant 1\}$: it is isomorphic to the Lie algebra of derivations of the free $\mathbb{C}$-Lie algebra L on d generators sending L to $[L, L]$.

Notation 3.1. Let G denote the free prounipotent group on $g_1, \ldots, g_d$ (1.1). Let $K_i = \mathrm{Ker}(\mathrm{Aut}(G) \to \mathrm{Aut}(G/C^{i+1}G))$ for $i = 0, 1, 2, \ldots$. Thus $K_0 = \mathrm{Aut}(G)$ and if $\sigma \in K_i$ for $i \geqslant 1$, then $g_p^{-1}\sigma(g_p) \in C^{i+1}G$.

We recall (1.10) that if $\sigma \in K_0$ there is an algebra automorphism s of $\mathbb{C}\langle\langle t_1, \ldots, t_d \rangle\rangle$ such that $s\theta(x) = \theta(\sigma(x))$ for all x in G, θ being the embedding (1.6).

Lemma 3.2. *In Notation* 3.1, *let $\sigma \in K_1$ and let s be the associated automorphism of $\mathbb{C}\langle\langle t_1, \ldots, t_d \rangle\rangle$. Then $\sigma \in K_r$ if and only if $s(t_p) - t_p \in J_{r+1}$* (1.7) *for $p = 1, \ldots, d$.*

Proof. Assume $\sigma \in K_r$ and let $a_p = g_p^{-1}\sigma(g_p)$ for $p = 1, \ldots, d$. Then $\theta(a_p) = (1 + t_p)^{-1} s(1 + t_p)$ is in $1 + J_{r+1}$ by (2.6); let $\theta(a_p) = 1 + b_p$ with $b_p J_{r+1}$. Then $(1 + t_p)(1 + b_p) = 1 + s(t_p)$ so $s(t_p) - t_p = (1 + t_p) b_p \in J_{r+1}$. If conversely $s(t_p) - t_p = f_p \in J_{r+1}$, then we let $\theta(g_p^{-1}\sigma(g_p)) = 1 + c_p$, and then $c_p = (1 + t_p)^{-1} f_p \in J_{r+1}$ so $g_p^{-1}\sigma(g_p) \in C^{r+1}G$ by (2.6).

If s is an algebra automorphism of $\mathbb{C}\langle\langle t_1, \ldots, t_d \rangle\rangle$ with $s(t_p) - t_p \in J_{r+1}$ for $p = 1, \ldots, d$, then if m is a monomial of degree i, $s(m) \equiv m(J_{r+i})$. Thus if $1 + x \in J_i$, $s(1 + x) \equiv (1 + x)(1 + J_{r+i})$. Applying (2.6) we obtain the following corollary of (3.2):

Corollary 3.3. *In Notation* 3.1, *let $\sigma \in K_r$ and let $a \in C^iG$. Then $a^{-1}\sigma(a) \in C^{r+i}G$.*

Lemma 3.4. *In Notation* 3.1, *let $\sigma \in K_r$ and $\tau \in K_s$. Then $(\sigma, \tau) g_p \equiv g_p$ (mod $C^{r+s+1}G$). In particular, we have $(K_r, K_s) \subseteq K_{r+s}$.*

Proof. Let $g_p^{-1}\sigma(g_p) = a \in C^{r+1}G$ and let $g_p^{-1}\tau(g_p) = b \in C^{s+1}G$. By (3.3), we know $\sigma(b) \equiv b(C^{r+s+1}G)$ and $\tau(a) \equiv a(C^{r+s+1}G)$. Thus, mod $C^{r+s+1}G$, we have $\sigma\tau(g_p) \equiv \sigma(g_p b) \equiv g_p ab \equiv g_p ba \equiv \tau(g_p a) \equiv \tau\sigma(g_p)$ (we used the fact that $(a, b) \in C^{r+s+1}$).

The groups $\{K_i\}$ form a central filtration of the group K_1. We next look at the computation of the successive quotients K_i/K_{i+1}. It is clear from the definition of K_i that we have a function $K_i \to (C^{i+1}G)^{(d)}$ which sends σ to $(g_1^{-1}\sigma(g_1),..., g_d^{-1}\sigma(g_d))$. This function is clearly injective. It is also surjective:

LEMMA 3.5. *In Notation* 3.1,

(1) *Let* $a_1,..., a_d \in C^{i+1}G$. *Then there is* $\sigma \in K_i$ *such that* $g_p^{-1}\sigma(g_p) = a_p$ *for* $p = 1,..., d$.

(2) $\mathrm{Aut}(G) \to \mathrm{Aut}(G/C^{i+1}G)$ *is surjective for all* i.

Proof. Since G is free on $g_1,..., g_d$, for any d elements $b_1,..., b_d$ of G, there is an endomorphism τ of G with $\tau(g_p) = b_p$. By [7, Corollary 2.6], τ is an isomorphism if and only if the induced endomorphism of G/C^2G is an isomorphism. Assertion (2) is now clear. For (1), we note that if $b_p = g_p a_p$, then τ is the identity on $G/C^{i+1}G$, and thus an automorphism of G in J_i.

The bijection $K_i \to (C^{i+1}G)^{(d)}$ is not a group homomorphism, but induces a group homomorphism on successive quotients, from which we can calculate the dimensions of the groups K_1/K_i.

THEOREM 3.6. *In Notation* 3.1, *the function* Φ_i: $K_i/K_{i+1} \to (C^{i+1}G/C^{i+2}G)^{(d)}$ *given by* $\Phi_i(\sigma K_{i+1}) = (g_1^{-1}\sigma(g_1)\, C^{i+2}G,..., g_d^{-1}\sigma(g_d)\, C^{i+2}G)$ *is an isomorphism of groups for* $i \geqslant 1$.

Proof. The bijection $K_i \to (C^{i+1}G)^{(d)}$ induces a surjection $K_i \to (C^{i+1}G/C^{i+1}G)^{(d)}$. Let $\sigma, \tau \in K_i$ with $\sigma(g_p) = g_p a_p$ and $\tau(g_p) = g_p b_p$. Then $(\sigma\tau)(g_p) = g_p a_p \sigma(b_p)$. Since $\tau \in K_i$, $b_p \in C^{i+1}G$ so $\sigma(b_p) \equiv b_p(C^{2i+1}G)$ by (3.3). Hence, modulo $C^{i+2}G$, we have $g_p^{-1}(\sigma\tau(g_p)) \equiv (g_p^{-1}\sigma(g_p))(g_p^{-1}(\tau(g_p))$ so our surjection is a homomorphism. The kernel consists of all σ such that $g_p^{-1}\sigma(g_p) \in C^{i+2}G$ for all p, which is K_{i+1}.

We note that (3.6) actually shows that K_1 is prounipotent: for it follows form (3.5)(2) that $K_1 = \mathrm{proj\,lim}\, K_1/K_i$ and from (3.6) that K_1/K_i is unipotent. It is further clear that each K_i is a Zariski-closed normal subgroup of K_1. We will elucidate the structure of K_1 below ((3.18)).

We now turn to the position of G inside K_1 as inner automorphisms.

Notation 3.7. Let G be as in (1.1) and K_1 as in (3.1). Let in: $G \to K_1$ be given by $\mathrm{in}(x)(y) = xyx^{-1}$. The image $\mathrm{in}(G)$ is the inner automorphism group of G, and elements of K_1 not in $\mathrm{in}(G)$ are outer automorphisms.

LEMMA 3.8. *In Notation* 3.7, in: $G \to K$, *is injective, and* $\text{in}^{-1}(K_i) = C^iG$ *for* $i \geqslant 1$.

Proof. By [7, Corollary 3.7], G has no center so in is injective. Next, we use the fact that the Lie algebra of the center of $G/C^{i+1}G$ is the center of $\text{Lie}(G/C^{i+1}G)$. By (2.3), this latter is $L/\mathscr{C}^{r+1}L$, where L is a free Lie algebra. By [4, Example 3, p. 186], the center of $L/\mathscr{C}^{r+1}L$ is $\mathscr{C}^rL/\mathscr{C}^{r+1}L$, and it follows that the center of $G/C^{i+1}G$ is $C^iG/C^{i+1}G$. If $x \in C^iG$, we conclude that $\text{in}(x) \in K_i$, while if $x \in G$ and $\text{in}(x) \in K_i$, then $xC^{i+1}G$ is central in $G/C^{i+1}G$, so x belongs to C^iG, and it follows that $\text{in}^{-1}(K_i) = C^iG$.

By combining (3.6) and (3.8) and the dimension formula for $C^rG/C^{r+1}G$ (2.8), we can establish the existence of outer automorphisms:

THEOREM 3.9. *Let G be a free prounipotent group on $d > 1$ generators. Then G has outer automorphisms which are trivial on G/C^2G.*

Proof. We need to see, in Notations 3.1 and 3.6, that $K_1 \neq \text{in}(G)$. Assume the contrary. By (3.8), we would have K_1/K_2 isomorphic to C^1G/C^2G, and hence by (2.8) of dimension $l_d(1) = d$, but by (3.6), K_1/K_2 is isomorphic $(C^2G/C^3G)^{(d)}$ so has dimension $dl_d(2)$. Thus $l_d(2) = 1$ which forces $d = 2$. Hence for $d > 2$, we have a contradiction. For $d = 2$, (3.7) implies that K_2/K_3 would be isomorphic to C^2G/C^3G, hence by (2.8) of dimension $l_2(2) = 1$, while (3.6) shows that K_2/K_3 is isomorphic to $(C^3G/C^4G)^{(2)}$ and so has dimension $2l_2(3) = 4$, which is again a contradiction.

The case $d = 2$ of (3.9) deserves special notice: we recall Nielsen's theorem that the discrete free group on two generators has no outer automorphisms which are the identity modulo the commutator subgroup. In this case ($d = 2$), let F be the discrete subgroup of G generated by $\{g_1, g_2\}$. Then from (2.7)(1), we have $C^2F = C^2G \cap F$. We have $\text{Aut}(F) \subseteq \text{Aut}(G)$ in the obvious way, and by Nielsen's theorem we conclude $\text{Aut}(F) \cap K_1 = \text{in}(F)$. Since $\text{in}(G)$ is a proper subgroup of K_1, we have that $\text{Aut}(F) \cap K_1$ is not Zariski-dense in K_1.

For d arbitrary, we let F still denote the discrete subgroup of G generated by $\{g_1, \ldots, g_d\}$. Since $\bigcap C^rG = \{e\}$, for example by (2.6), we know from [7, (1.5)] that $G = \text{proj lim}\{G/C^rG\}$, so by (2.7), G also contains $\bar{F} = \text{proj lim}\{F/C^rF\}$. It will follow from (3.11) and [1, Theorem 2.3, p. 242] that $\text{Aut}(G)$ contains $\overline{\text{Aut}(F)} = \text{proj lim Aut}(F/C^rF)$, and we will show that $\overline{\text{Aut}(F)} \cap K_i$ is Zariski-dense in K_i.

Notation 3.10. Let G denote the free prounipotent group on $g_1, \ldots, g_d$, let F be the discrete subgroup of G generated by $\{g_1, \ldots, g_d\}$, let $G(r) = G/C^{r+1}G$ and let $F(r) = F/C^{r+1}F$. (By (2.7)(1), $F(r)$ is the subgroup of $G(r)$ generated by the images $\bar{g}_1, \ldots, \bar{g}_d$ of the g_p's.) Let $K_0(r) = \text{Aut}(G(r))$ and let $K_i(r) =$

$\mathrm{Ker}(K_0(r) \to \mathrm{Aut}(G(r)/C^{i+1}G(r))$. (For $i \leqslant r$, $G(r)/C^{i+1}G(r) = G(i)$ and using (3.4)(2) we have in Notation 3.1 that $K_i(r) = K_i/K_r$.) Let $L_0(r) = \mathrm{Aut}(F(r))$ and $L_i(r) = \mathrm{Ker}(L_0(r) \to \mathrm{Aut}(F(r)/C^{i+1}F(r))$. (For $i \leqslant r$, $F(r)/C^{i+1}F(r) = F(i)$, so $L_i(r) = \mathrm{Ker}(L_0(r) \to L_0(i)$.) Let $L_i = \mathrm{proj\,lim}\{L_i(r) \mid r \geqslant i\}$. (It follows from [1, Theorem 2.3] that L_0 is a group of automorphisms of $\bar{F}$ and that L_i is the group denoted by the same symbol in [1, p. 242]).

Our notation is chosen to conform to [1]. We caution the reader that L_i is not a Lie algebra.

LEMMA 3.11. *In Notations* 3.1 *and* 3.10, *let* p_r: $K_1 \to K_1(r)$ and p_r^s: $K_1(s) \to K_1(r)$ $(s \geqslant r)$ *be the projections. Then the maps* p_r *induce an isomorphism* p: $K_0 \to \mathrm{proj\,lim}\{(K_0(r), p_r^s) \mid r \geqslant 1\}$. *Moreover*:

(1) $p(K_i) = \mathrm{proj\,lim}\{(K_i(r), p_r^s) \mid r \geqslant i\}$.

(2) *There are injections* $L_0(r) \to K_0(r)$ *for each* r *which induce an injection* $L_0 \to K_0$, *and* $K_i \cap L_0 = L_i$.

Proof. Let $\mathscr{S} = \{K_r \mid r \geqslant 1\}$. Then $K_r = \mathrm{Ker}(p_r)$ and $\bigcap \{H \mid H \in \mathbb{S}\} = e$. It follows from [7, (1.5)] that $K_0 = \mathrm{proj\,lim}\{K_0/H \mid H \in \mathscr{S}\}$. A similar argument proves (1): we replace $\mathscr{S}$ by $\{K_r \mid r \geqslant i\}$ and use the fact that for $i \leqslant r$, $p_r(K_i) = K_i/K_r = K_i(r)$. Let $\bar{\sigma}$ be an endomorphism of $F(r)$, and let σ be an endomorphism of F inducing $\bar{\sigma}$. Then σ extends to an endomorphism s of G, and the induced endomorphism $\bar{s}$ of $G(r)$ extends $\bar{\sigma}$. If σ is an automorphism, $\bar{s}$ is onto since $F(r)$ is Zariski-dense in $G(r)$, hence $\bar{s}$ is an automorphism. Then density of $F(r)$ in $G(r)$ guarantees the uniqueness of $\bar{s}$, given $\bar{\sigma}$. Thus we have an injection $L_0(r) \to K_0(r)$. The definition of L_0 and the identification of K_0 as an inverse limit shows that the induced injection on inverse limits embeds L_0 in K_0. Since $C^{r+1}G \cap F = C^{r+1}F$ (2.7)(1), we have $K_i \cap L_0 = L_i$.

Now we establish our density result:

THEOREM 3.12. *In Notations* 3.1 *and* 3.10, L_i *is Zariski-dense in* K_i *for* $i \geqslant 1$. *Moreover,* L_i/L_{i+1} *is a discrete free abelian group with* $L_i/L_{i+1} \otimes \mathbb{C} = K_i/K_{i+1}$.

Proof. For $r \geqslant i$, it follows from [1, p. 242] that $L_i/L_r = L_i(r)$. We first show that $L_i(i+1)$ is Zariski-dense in $K_i(i+1)$ (they are embedded by (3.11)(2). The function Φ_i: $K_i/K_{i+1} \to (C^{i+1}G/C^{i+2}G)^{(d)}$ of (3.6) is an isomorphism of complex vector spaces and $\Phi_i(L_i/L_{i+1}) = (C^{i+1}F/C^{i+2}F)^{(d)}$ by [1, p. 246]. It follows from (2.7) that $\Phi_i(L_i/L_{i+1})$ is Zariski-dense in $\Phi_i(K_i/K_{i+1})$ and that $\Phi_i(L_i/L_{i+1}) \otimes \mathbb{C} = \Phi_i(K_i/K_{i+1})$. Thus $L_i(i+1)$ is Zariski-dense in $K_i(i+1)$. For $r > i$, the exact sequence $1 \to K_{r-1}(r) \to K_i(r) \to K_i(r-1) \to 1$, $1 \to L_{r-1}(r) \to L_i(r) \to L_i(r-1) \to 1$ and induction on

r show that $L_i(r)$ is Zariski-dense in $K_i(r)$. Let $\bar{K}_i$ be the Zariski-closure of L_i in K_i. Then, in the notation of (3.11), we have $p_r(\bar{K}_i) = K_i(r)$ for $r \geqslant i$; since $K_i = \text{proj lim}\{K_i(r) \mid r \geqslant i\}$ by (3.11)(1), we conclude that $\bar{K}_i = K_i$.

The maps Φ_i of (3.6) induce a vector space isomorphism Φ: $\bigoplus_{\prime}\{K_i/K_{i+1} \mid i \geqslant 1\}$ $[\mathscr{C}^2(\text{gr}(G))]^{(d)}$. Now $\{K_i\}$ is a central filtration of K_1, so the domain of Φ is a Lie algebra, but Φ is not a Lie algebra isomorphism. Our next task will be to identify the Lie algebra $\bigoplus K_i/K_{i+1}$.

Definition 3.13. Let L be a Lie algebra $\text{Der}_i(L) = \{D \in \text{Der}(L) \mid D(L) \subset \mathscr{C}^{i+1}L\}$ for $i \geqslant 0$.

We note the following properties of (3.13), whose elementary proofs are omitted:

Lemma 3.14. *Let L be a Lie algebra.*

(1) *If $D \in \text{Der}_i(L)$, $D(\mathscr{C}^pL) \subseteq \mathscr{C}^{p+i}L$.*

(2) $[\text{Der}_i(L), \text{Der}_j(L)] \subseteq \text{Der}_{i+j}(L)$.

(3) $\{\text{Der}_i(L) \mid i \geqslant 0\}$ *is a descending chain of ideals of* $\text{Der}_0(L) = \text{Der}(L)$.

Now $\text{gr}(G)$ is, by (2.7), the free Lie algebra on $\{\bar{g}_p = g_p C^2G\}$. By [4, Corollary, p. 130] we have an isomorphism of vector spaces $\text{Der}_i(\text{gr}(G)) \to [\mathscr{C}^{i+1}(\text{gr}(G))]^{(d)}$ which sends the derivation D to the d-tuple whose pth entry is $D(g_p)$. Thus we obtain, via the map Φ defined after (3.12), a vector space isomorphism $\bigoplus K_i/K_{i+1} \to \text{Der}_1(\text{gr}(G))$ which sends σK_{i+1} to D_σ where $D_\sigma(\bar{g}_p) = g_p^{-1}\sigma(g_p)\, C^{i+2}G$. We want to prove that $\sigma \to D_\sigma$ is a Lie algebra isomorphism. For the purposes of that proof, we give the following notation:

Notation 3.15. In Notation 3.10, let $\mathscr{G} = \text{gr}(G)$ and $\mathscr{G}(r) = \text{gr}(G(r))$. Let $K = \bigoplus\{K_i/K_{i+1} \mid i \geqslant 1\}$ and $K(r) = \bigoplus\{K_i(r)/K_{i+1}(r) \mid 1 \leqslant i \leqslant r-1\}$. We use x_p to denote $g_p C^2G$ in $\mathscr{G}$ and $\bar{x}_p$ to denote $\bar{g}_p C^2G(r)$ in $\mathscr{G}(r)$.

By (2.3) and (2.7) we have an isomorphism $\mathscr{G}(r) \to \text{Lie}(G(r))$ such that the exponential of the image of $\bar{x}_p$ is $\bar{g}_p$. Let $E: \mathscr{G}(r) \to G(r)$ be the composition of this isomorphism with the exponential map. Since $G(r)$ is unipotent, we can identify $\text{Aut}(G(r))$ with $\text{Aut}(\text{Lie}(G(r)) = \text{Aut}(\mathscr{G}(r))$, and the latter has Lie algebra $\text{Der}(\mathscr{G}(r))$. Thus $\text{Der}(\mathscr{G}(r))$ is isomorphic to $\text{Lie}(K_0(r))$; let $\text{Ex}: \text{Der}(\mathscr{G}(r)) \to K_0(r)$ be the composite of this isomorphism with the exponential map for $K_0(r)$. We claim for $i \geqslant 1$, $\text{Ex}(\text{Der}_i(\mathscr{G}(r)) = K_i(r)$: for if $D \in \text{Der}(\mathscr{G}(r))$, and $X \in \mathscr{G}(r)$, we have $\text{Ex}(D)(E(X)) = E(e^DX)$. Suppose $D \in \text{Der}_i(\mathscr{G}(r))$, $i \geqslant 1$. We need to prove that $\text{Ex}(D)(\bar{g}_p) \equiv \bar{g}_p$ (mod $C^{i+1}G(r)$). Now $\bar{g}_p = E(\bar{x}_p)$, and $e^D\bar{x}_p = \bar{x}_p + h$ where $h \in \mathbb{C}^{i+1}\mathscr{G}(r)$. E commutes with the projections $\mathscr{G}(r) \to \mathscr{G}(i)$ and $G(r) \to G(i)$, so we have $E(\bar{x}_p + h) \equiv E(\bar{x}_p)$ (mod $C^{i+1}G(r)$), so obtain the desired congruence on

$\mathrm{Ex}(D)$. This proves that the analytic subgroup $\bar{K}_i(r)$ of $K_0(r)$ with Lie algebra $\mathrm{Der}_i(\mathscr{G}(r))$ is contained in $K_i(r)$. On the other hand, it follows from the isomorphism (of vector spaces) $\mathrm{Der}_i(\mathscr{G}(r)) \to [\mathscr{C}^{i+1}\mathscr{G}(r)]^{(d)}$ and (3.6) that $K_i(r)$ and $\bar{K}_i(r)$ have the same dimensions, so they are equal. Now since $K_i(r)$ is unipotent for $i \geqslant 1$, we have $\mathrm{Ex}(\mathrm{Der}_i(\mathscr{G}(r)) = K_i(r)$. In particular, we conclude:

LEMMA 3.16. *In Notation* 3.15 *for* $i \geqslant 1$, *there is an isomorphism* $\mathrm{Der}_i(\mathscr{G}(r)) \to \mathrm{Lie}(K_i(r))$ *whose composition with the exponential map gives the analytic bijection* $\mathrm{Ex}\colon \mathrm{Der}_i(\mathscr{G}(r)) \to K_i(r)$.

We use (3.16) to identify $\mathrm{Der}_i(\mathscr{G}(r))$ with $\mathrm{Lie}(K_i(r))$. Now the unipotent group $K_1(r)$ has $\{K_i(r)\}$ as a central filtration, and the associated graded vector space is a Lie algebra. We want to relate this Lie algebra to $\mathrm{Lie}(K_1(r)) = \mathrm{Der}_1(\mathscr{G}(r))$.

LEMMA 3.17. *Let* H *be a unipotent group with Lie algebra* L. *Let* $\{H_i\}$ *be a central filtration of* $H_1 = H$ *by unipotent subgroups and let* $L_i = \mathrm{Lie}(H_i)$. *Then* $\{L_i\}$ *is a filtration of* $L_1 = L$ *and* $\exp\colon L \to H$ *induces a Lie algebra isomorphism* $\mathrm{Exp} = \mathrm{gr}(\exp)\colon \mathrm{gr}(L) \to \mathrm{gr}(H)$ *of associated graded Lie algebras, where* $\mathrm{Exp}(X + L_{i+1}) = \exp(X)\, H_{i+1}$ *for* $X \in L_i$.

Proof. It is clear that Exp is an isomorphism of complex vector spaces. We need to see that it is a Lie algebra isomorphism. Let $x = X + L_{i+1}$ and $y = Y + L_{j+1}$ be elements of L_i/L_{i+1} and L_j/L_{j+1}. Then $[x, y] = [X, Y] + L_{i+j+1}$ in $\mathrm{gr}(L)$, and $\mathrm{Exp}[x, y] = \exp[X, Y]\, H_{i+j+1}$ in $\mathrm{gr}(H)$. So we need to show that $\exp[X, Y] \equiv (\exp X, \exp Y) \pmod{H_{i+j+1}}$. We can assume, by passing to quotient, that $H_{i+j+1} = 1$ and $L_{i+j+1} = 0$, so that L_{i+j} (and hence $[X, Y]$) is central in L. Then $(\exp X, \exp Y) = ((\exp X)^{-1}, (\exp Y)^{-1}) = \exp(X) \exp(Y) \exp(X)^{-1} \exp(Y)^{-1} = \exp(\mathrm{Ad}(\exp X)(Y)) \exp(Y)^{-1} = \exp(\exp(\mathrm{ad}\, X)\, Y) \exp(Y)^{-1}$. Since $[X, Y]$ is central, $\exp(\mathrm{ad}\, X)\, Y = Y + [X, Y]$, and Y commutes with $[X, Y]$. So $\exp(\exp(\mathrm{ad}\, X)\, Y) = \exp[X, Y] \exp Y$ and $(\exp X, \exp Y) = \exp[X, Y]$.

We apply (3.17) to $K_1(r)$ and the filtration $\{K_i(v)\}$. Taking (3.16) into account, we have a Lie algebra isomorphism $\mathrm{gr}(\mathrm{Der}_1(\mathscr{G}(r))) \to \mathrm{gr}(K_1(r))$ which sends $D + \mathrm{Der}_{i+1}(\mathscr{G}(r))$ to $\mathrm{Ex}(D)\, K_{i+1}(r)$ for $D \in \mathrm{Der}_i((r))$. Now $\mathrm{Der}_1(\mathscr{G}(r))$ is graded: let $\mathrm{Der}^i(\mathscr{G}(r)) = \{D \in \mathrm{Der}_1(\mathscr{G}(r)) \mid D(\bar{x}_p) \in C^{i+1}G(r)/C^{i+2}G(r)$ for $1 \leqslant p \leqslant d\}$. It is clear that $\mathrm{Der}_1 = \bigoplus\{\mathrm{Der}^i \mid i \geqslant 1\}$ and since $\mathscr{G}$ is free we also conclude $[\mathrm{Der}^i, \mathrm{Der}^j] \subseteq \mathrm{Der}^{i+j}$. Thus we have a Lie algebra isomorphism $\mathrm{Der}_1(\mathscr{G}(r)) \to \mathrm{gr}(\mathrm{Der}_1(\mathscr{G}(r)))$ which sends D in Der^i to $D + \mathrm{Der}_{i+1}$.

Composing the above isomorphisms, we find a Lie algebra isomorphism $\mathrm{Der}_1(\mathscr{G}(r)) \to \mathrm{gr}(K_1(r))$ which sends D in $\mathrm{Der}^i(\mathscr{G}(r))$ to $\mathrm{Ex}(D)\, K_{i+1}(r)$. We recall that we also have a linear isomorphism $\mathrm{gr}(K_1(r)) \to \mathrm{Der}_1(\mathscr{G}(r))$ which sends σK_{i+1}, for $\sigma \in K_i$, to the derivation sending $\bar{x}_p$ to $\bar{g}_p^{-1}\sigma(\bar{g}_p)\, C^{i+2}G(r)$. We claim that the composite of these two maps is the identity: for let $D \in \mathrm{Der}^i(\mathscr{G}(r))$. We must show that $D(\bar{x}_p) \equiv \bar{g}_p^{-1}\,\mathrm{Ex}(D)\,\bar{g}_p \pmod{C^{i+2}G(r)}$. We can pass to $G(i+1)$ (so that $C^{i+2}G(i+1) = e$). Then $\mathrm{Ex}(D)(\bar{g}_p) = \mathrm{Ex}(D)(E(\bar{x}_p)) = E(e^D \bar{x}_p)$. Now $D(\bar{x}_p) \in \mathscr{C}^{i+1}\mathscr{G}(i+1)$ so $D(\bar{x}_p)$ is central and $D^2(\bar{x}_p) = 0$. Thus $e^D\bar{x}_p = \bar{x}_p + D(\bar{x}_p)$, and $\bar{x}_p$ and $D(\bar{x}_p)$ commute so $E(\bar{x}_p + D(\bar{x}_p)) = E(\bar{x}_p)\,E(D(\bar{x}_p)) = \bar{g}_p E(D(\bar{x}_p))$. Thus $\bar{g}_p^{-1}\,\mathrm{Ex}(D)\,\bar{g}_p = E(D(\bar{x}_p))$. Now $\mathscr{C}^{i+1}\mathscr{G}(i+1) = C^{i+1}G(i+1)$ and, under our isomorphism $\mathscr{G}(i+1) \to \mathrm{Lie}(G(i+1))$, E becomes the identity on $\mathscr{C}^{i+1}\mathscr{G}(i+1)$, so $E(D(\bar{x}_p)) = D(\bar{x}_p)$. This means that our composite is the identity on $\mathrm{Der}^i(\mathscr{G}(r))$, and hence is the identity on $\mathrm{Der}_1(\mathscr{G}(r))$.

In particular, we have prove that $\mathrm{gr}(K_1(r)) \to \mathrm{Der}_1(\mathscr{G}(r))$ is a Lie algebra isomorphism.

We can now identify the Lie algebra $\mathrm{gr}(K_1)$.

THEOREM 3.18. *In Notation 3.1, let* Φ: $\bigoplus \{K_i/K_{i+1} \mid i \geqslant 1\} \to \mathrm{Der}_1(\mathrm{gr}(G))$ *be given by* $\Phi(\sigma K_{i+1})(g_p C^2 G) = g_p^{-1}\sigma(g_p)\, C^{i+2}G$. *Then* Φ *is a Lie algebra isomorphism.*

Proof. $\mathrm{Der}_1(\mathrm{gr}(G)) = [\mathscr{C}^2\,\mathrm{gr}(G)]^{(d)} = \bigoplus\{(C^{i+1}G/C^{i+2}G)^{(d)} \mid i \geqslant 1\}$ and so by (3.6), Φ is a vector space isomorphism. As noted above, for each r, $\mathrm{gr}(K_1(r)) \to \mathrm{Der}_1(\mathscr{G}(r))$ is a Lie algebra isomorphism, and passing to the projective limit gives a Lie algebra isomorphism $\hat{\Phi}$: $\mathrm{Proj\,lim}\{\mathrm{gr}(K_1(r)\} \to \mathrm{proj\,lim}\{\mathrm{Der}_1(\mathscr{G}(r)\}$. Now $\mathrm{gr}(K_1)$ embeds in the domain of $\hat{\Phi}$, $\mathrm{Der}_1(\mathrm{gr}(G))$ embeds in the range of $\hat{\Phi}$ and $\hat{\Phi}$ restricts to Φ on $\mathrm{gr}(K_1)$, so Φ is also a Lie algebra isomorphism.

4. Some Abstract Subgroups of Free Prounipotent Groups

In this section we consider some abstract subgroups of the free prounipotent group G on generators $g_1, \dots, g_d$. We first show that the one parameter subgroups through the generators generate a group isomorphic to a free product of d copies of $\mathbb{C}$. This implies that G contains Zariski-dense subgroups isomorphic to d-fold free products of large abelian groups. The d-fold free product, $P(L)$, of d copies of the additive group of any subfield of $\mathbb{C}$ is also embedded in G. We find that $P(L)/C^nP(L)$ is torsion free, but in general $C^nP(L) \neq P(L) \cap C^nG$, although equality occurs when $L = \mathbb{Q}$.

We further see that two generator subgroups of G are abelian or free, and that a Zariski-dense subgroup Γ with $\Gamma/C^2\Gamma$ d-generated is d-para free. For such a Γ, any homomorphism from Γ to a prounipotent group extends uniquely to G. When $d=2$, we exhibit a Zariski-dense free subgroup of rank three contained in no rank 2 free subgroup.

If H is a prounipotent group and h an element of H, then there is a unique homomorphism $f\colon G_a \to H$ with $f(1)=h$ [7, Lemma 3.4]. We identify G_a with $\mathbb{C}$ as additive groups and if $a \in \mathbb{C}$ we write h^a for $f(a)$. It is then clear that $h^a h^b = h^{a+b}$, $(h^a)^b = h^{ab}$, and h^a is the usual power for $a \in \mathbb{Z}$. If $H=G$ is the free prounipotent group of (1.1), and θ is the embedding of (1.6), then if $\theta(h)=1+f$, $\theta(h^a) \doteq (1+f)^a = \sum \binom{a}{i} f^i$ where $\binom{a}{i} = (i!)^{-1}(a(a-1)\cdots(a-i+1))$. In this case, we will be interested in the abstract subgroup of G generated by all powers of the generators g_p of G:

Proposition 4.1. *Let G be the free prounipotent group on $g_1,\ldots, g_d$, let L be a subring of $\mathbb{C}$ and let $P(L)$ be the abstract subgroup of G generated by $\{g_p^a \mid a \in L,\ 1 \leqslant p \leqslant d\}$. Then $P(L)$ is a d-fold free product of copies of the additive group of L.*

Proof. We have an obvious surjection of the free product to $P(L)$. Suppose $a_1,\ldots, a_n \in L - \{0\}$, $z_1,\ldots, z_n$ are in $\{g_p\}$ with $z_{i+1} \neq z_i$ for all i, and $z_1^{a_1} \cdots z_n^{a_n} = e$. We apply the embedding θ of (1.6) to the product: let $\theta(z_i) = 1 + x_i$, where $x_i \in \{t_1,\ldots, t_d\}$. By the above remarks, we have $\prod (1+x_i)^{a_i} = 1$. The coefficient of $x_1 \cdots x_n$ on the left-hand side is $a_1 \cdots a_n$, which is not zero, and hence there is no such product. It follows that the surjection from the free product to $P(L)$ is an isomorphism.

We observe that $P(F)$ is Zariski-dense in G. We thus obtain some large subgroups of G:

Corollary 4.2. *Let $\{A_i \mid i \leqslant i \leqslant d\}$ be a set of non-trivial torsion-free abelian groups of rank at most the cardinality of $\mathbb{C}$, and let $A = A_1 * \cdots * A_d$. Then A can be embedded as a Zariski-dense subgroup of the free prounipotent group on d generators.*

Proof. Embed A in $P(\mathbb{C})$ by (4.1).

Let G be as in (1.1) and $G(r) = G/C^{r+1}G$ as in (2.1). By [6, Proposition, p. 55], the natural map $P(\mathbb{C}) \to G(r)$ is surjective, and its kernel contains $C^{r+1}P(\mathbb{C})$. We now see that this containment is in general proper.

Example 4.3. In Notations 1.1 and 2.1, assume $d=2$ and $r=2$. Let $P = P(\mathbb{C})$. Then $C^3 G \cap P$ properly contains $C^3 P$: for we may identify $G(r)$ with the matrix group

$$U = \left\{ M(a, g, x) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 1 & b & 1 \end{pmatrix} \quad a, b, x \in \mathbb{C} \right\}.$$

Let $\phi\colon \mathbb{C} \to \mathbb{Q}$ be a rational linear function with $\phi(1) = 1$. Then $\phi_i\colon \mathbb{C} \to U$ by $\phi_1(a) = M(\phi(a), 0, 0)$ and $\phi_2(b) = M(0, \phi, (b), 0)$ are group homomorphisms, which induces a homomorphism $\phi_1 * \phi_2$ from $\mathbb{C} * \mathbb{C} = P$ to U. If $C^3G \cap P = C^3P$, then $\phi_1 * \phi_2$ induces a group homomorphism $\Phi\colon U \to U$ (the domain U is $P/C^3G \cap P = G(2)$), with $\Phi(M(a, 0, 0)) = \phi_1(a)$ and $\Phi(M(0, b, 0)) = \phi_2(b)$. Since $(M(x, 0, 0), M(0, -1, 0)) = M(0, 0, x)$, we have $\Phi(M(0, 0, x)) = (\phi_1(a), \phi_2(a)) = M(0, 0, \phi(x))$, and since $M(a, b, c) = M(a, 0, 0,)\, M(a, b, 0)\, M(0, 0, x)$ we have $\Phi(M(a, b, x)) = M(\phi(a), \phi(b), \phi(x))$. Now $(M(0, b, 0), M(a, 0, 0)) = M(a, b, ab)$, so it follows that $\Phi(M(a, b, ab)) = M(\phi(a), \phi(b), \phi(a)\,\phi(b))$ and $\Phi(M(a, b, ab)) = M(\phi(a), \phi(b), \phi(ab))$ so $\phi(ab) = \phi(a)\,\phi(b)$. Of course, there is no such functional ϕ.

Even though, as (4.3) shows, we have $P(\mathbb{C})/C^{r+1}P(\mathbb{C}) \neq G/C^{r+1}G$, we can still say something about the first quotient.

LEMMA 4.4. *Let L be a field of characteristic zero and let P be the d-fold free product of copies of the additive group of L. Then* $\operatorname{gr}(P)$ *is a rational vector space. In particular, P/C^rP is a torsion free nilpotent group.*

Proof. Let $a \in \mathbb{Z}$ be non-zero and let s be the automorphism of P which is multiplication by a on each factor of L in the product P. Then s_a induces an automorphism t_a of $\operatorname{gr}(P)$. Now $\operatorname{gr}(P)$ is generated by P/C^2P as a Lie algebra over $\mathbb{Z}$, and on $P/C^2P = F \times \cdots \times F$, t_a is just multiplication by a. Thus on $C^rP/C^{r+1}P$, t_a is multiplication by a^r. This is an automorphism, so $C^rP/C^{r+1}P$ is a^r-torsion-free divisible, and it follows that $\operatorname{gr}(P)$ is a rational vector space.

By (4.4), $\operatorname{gr}(P)$ is a rational Lie algebra. But it is not, in general, an L algebra: if it were, in the case $L = \mathbb{C}$, then $f\colon \operatorname{gr}(P(\mathbb{C})) \to \operatorname{gr}(G)$ would be a homomorphism of $\mathbb{C}$-Lie algebras (notation as in (4.1)). Since $P(\mathbb{C})/C^2P(\mathbb{C}) = G/C^2G, f$ would be surjective, and since $\operatorname{gr}(G)$ is free by (2.7), it would be an isomorphism. This implies that $C^rG \cap P(\mathbb{C}) = C^rP(\mathbb{C})$ for all r, contrary to (4.3).

We can say something if $L = \mathbb{Q}$:

PROPOSITION 4.5. *In Notation* 1.1 *and* 4.1, $C^rG \cap P(\mathbb{Q}) = C^rP(\mathbb{Q})$ *for all r.*

Proof. Consider the inclusions $F = P(\mathbb{Z}) \subseteq P = P(\mathbb{Q}) \subseteq G$. These induce maps $\operatorname{gr}(F) \to \operatorname{gr}(P) \to \operatorname{gr}(G)$, and by (4.4) these induce maps $\operatorname{gr}(F) \otimes \mathbb{Q} \to \operatorname{gr}(P) \to \operatorname{gr}(G)$ of $\mathbb{Q}$-Lie algebras. The first of these is surjective since $F/C^2F \otimes \mathbb{Q} = P/C^2P$ and the composite is injective by (2.7), so $\operatorname{gr}(P) \to \operatorname{gr}(G)$ is injective, and the result follows.

It follows from (4.5) that when $L = \mathbb{Q}$, the torsion free nilpotent group P/C^rP of (4.4) has a faithful unipotent representation. We do not know if this is true for any other subfields of $\mathbb{C}$.

We turn to finitely generated abstract sugroups of the free prounipotent group G. If Γ is such a subgroup, its Zariski-closure is again a free unipotent group by [7, Corollary 2.10], so we may assume Γ is Zariski-dense in G. We begin this study in the case Γ is two-generated.

LEMMA 4.6. *Let G be as in* (1.1) *and let Γ be Zariski-dense abstract subgroup of G. Then Γ contains elements x_p, $1 \leqslant p \leqslant d$, such that G is free prounipotent on $\{x_p \mid 1 \leqslant p \leqslant d\}$. In particular, Γ contains a free subgroup of rank d which is Zariski-dense in G.*

Proof. The image of Γ is Zariski-dense in G/C^2G, hence spans this complex vector space over C. Choose $x_1,..., x_d$ in Γ such that $\{x_p C^2G\}$ is a basis of G/C^2G. There is an automorphism s of G with $s(g_p) = x_p$ $1 \leqslant p \leqslant d$, so $\{x_p\}$ generates a free subgroup F of G, and Z is Zariski-dense in G.

COROLLARY 4.7. *Let Γ be a two-generated subgroup of a free prounipotent group G. Then either*

(1) *Γ is abelian and there is $z \in G$ and $a, b \in \mathbb{C}$ with Γ generated by z^a, z^b, or*

(2) *Γ is free.*

Proof. We may assume Γ is Zariski-dense in G and hence G is two-generated. If Γ is abelian, so is G so $G = \mathbb{C}$ and (1) follows. If Γ is not abelian, and Γ is generated by x and y, then their images are a basis of G/C^2G so as above x, y generate a free subgroup of G.

Next, we identify certain parafree subgroups of G.

A group Γ is d-parafree if it contains a free subgroup F on d generators with $\mathrm{gr}(F) = \mathrm{gr}(\Gamma)$. (See [2, 3].)

PROPOSITION 4.8. *Let G be free prounipotent on d generators and let Γ be a Zariski-dense subgroup of G. Then Γ is d-parafree if and only if $\Gamma/C^2\Gamma$ is d-generated.*

Proof. Assume $\Gamma/C^2\Gamma$ is d-generated. Then the image $\bar{\Gamma}$ of Γ in G/C^2G is free abelian of rank at most d, and spans over $\mathbb{C}$, so $\bar{\Gamma}$ is free abelian of rank d and a quotient of $\Gamma/C^2\Gamma$. We can choose elements $x_1,..., x_d$ in Γ such that $\{x_p C^2\Gamma\}$ is an $\mathbb{Z}$-basis of $\Gamma/C^2\Gamma$ and $\{x_p C^2G\}$ is a $\mathbb{C}$-basis G/C^2G. Let F be the subgroup of Γ generated by $\{x_p\}$. F is free, and G is free prounipotent on x_p. $\mathrm{gr}(F) \to \mathrm{gr}(\Gamma)$ is then onto (since $F/C^2F \to \Gamma/C^2\Gamma$ is an isomorphism), and the composite $\mathrm{gr}(F) \to \mathrm{gr}(\Gamma) \to \mathrm{gr}(G)$ is injective by (2.7), so $\mathrm{gr}(F) \to \mathrm{gr}(\Gamma)$ is an isomorphism.

If Γ is d-parafree we can always find a free subgroup F of Γ with $\mathrm{gr}(F) \to \mathrm{gr}(\Gamma)$ an isomorphism. Suppose Γ is a Zariski-dense subgroup of the free prounipotent group G, so that G is free on the generators of F. Let f: $\Gamma \to U$ be any homomorphism from Γ to a unipotent group U. Since $C^nU = e$ for some n, f factors through $\Gamma/C^n\Gamma = F/C^nF$. Since f restricted to F has a unique extension to G, it follows that f extends uniquely to G. If the prounipotent group H is the inverse limit of unipotent groups $\{U_i \mid i \in I\}$, we have $\mathrm{Hom}(\Gamma, H) = \mathrm{proj\,lim}\ \mathrm{Hom}(\Gamma, U_i) = \mathrm{proj\,lim}\ \mathrm{Hom}(G, U_i) = \mathrm{Hom}(G, H)$ (the first two "hom's" are in the abstract sense and the latter in the unipotent sense). We have established:

PROPOSITION 4.9. *Let G be free prounipotent on d generators and let Γ be a Zariski-dense d-parafree subgroup of G. Then any homomorphism from Γ to a prounipotent group extends uniquely to a homomorphism with domain G.*

Because of (4.9), we can regard G as being the most general prounipotent closure of the group Γ. As we noted prior to (3.10), G contains $\bar{F} = \mathrm{proj\,lim}\{F/C^rF\}$, so by [3, Theorem 4.1, p. 231], G contains every parafree Γ with $\mathrm{gr}(F) = \mathrm{gr}(\Gamma)$. Not all these are free (even if finitely generated) [2, Theorem 1, p. 210], so we see that the prounipotent closure of Γ can be free prounipotent without Γ being free.

If G is free prounipotent on $g_1, \ldots, g_d$ and F is the free subgroup generated by $\{g_p\}$, any subgroup Γ of G of finite index is Zariski-dense in G, and free of finite rank. Thus G contains Zariski-dense free subgroups of arbitrary large rank, but which are subgroup of rank d free groups. The following example shows that there are also Zariski-dense free subgroups contained in no rank d free subgroup.

EXAMPLE 4.10. Let G be free prouipotent on two generators g, h, and let $\alpha \in \mathbb{C}$ be a root of no non-zero degree two rational polynomial. Then the subgroup Γ of G generated by g, h, $(gh)^\alpha$ is free of rank three, Zariski-dense in G, and contained in no rank two free subgroup of G. For let $z = (gh)^\alpha$, and let a_i, b_i, c_i, $1 \leqslant i \leqslant n$, be integers such that $u = \prod g^{a_i}h^{b_i}z^{c_i} = e$. Let θ: $G \to \mathbb{C}\langle\langle x, y\rangle\rangle$ be an embedding as in (1.6) such that $\theta(g) = 1 + x$ and $\theta(h) = 1 + y$. Then $\theta(u) = \prod (1 + x)^{a_i}\ (1 + y)^{b_i}\ ((1 + x)(1 + y))^{\alpha c_i}$ and $\theta(u) = 1$. The coefficient of xy in this product is $\sum a_i b_j + \sum a_i(\alpha c_j) + \sum \alpha c_i + \sum \binom{\alpha c_i}{2} = r\alpha^2 + s\alpha + t$ where $r = \frac{1}{2} \sum c_i^2$ and $s, t \in \mathbb{Q}$. Since r, s, t are rational, and $r\alpha^2 + s\alpha + t = 0$, our assumption on α implies that $r = 0$. Thus $\sum c_i^2 = 0$ so $c_1 = c_2 = \cdots = c_n = 0$. Thus u contains no factors of z, and since g, h generate a free group we conclude that Γ is free. The image of Γ in G/C^2G is free abelian of rank three, and it follows that Γ is not contained in any rank two free subgroup of G.

5. The Malcev Embedding

Malcev [12], using the theory of real nilpotent Lie groups and nil-manifold, discovered that finitely-generated torsion free nilpotent discrete groups have faithful representations in (real) unipotent groups. This "Malcev embedding" has been given various proofs: by Jennings [11] using group algebras (an exposition is given in [10, Chap. 4]) and by Warfield [15], following Quillen [13], using primitive elements for complete group algebras. Another version, using real algebraic groups is given by Ragunathan [14, Theorem 2.18, p. 40].

This embedding is also a consequence of our work here on the associated graded Lie algebra of a free prounipotent group, as we show in this section. In keeping with the expository character of this section, we present a self-contained proof, but we have not included the collateral parts of discrete group theory (as available, for example, in [10]) even though they also follow from our methods, as the interested reader can easily establish.

It is also an easy consequence of the Malcev embedding to see that discrete groups which are residually finitely generated torsion free nilpotent are embeddable in prounipotent groups. We have not dealt with that observation and its consequences here. However, we feel that prounipotent group theory is useful in the study of such groups, especially for the parafree groups, which embed in free unipotent groups, and we intend to return to this theory in a subsequent paper.

We begin by establishing some notation to be used throughout this section.

Notation 5.1. G is the free unipotent group on $g_1,\dots, g_d$. $G(\mathbb{Q})$ is the subgroup of G generated by $\{g_p^a \mid a \in \mathbb{Q},\ 1 \leqslant p \leqslant d\}$ and $G(\mathbb{Z})$ is the subgroup of G generated by $\{g_p \mid 1 \leqslant p \leqslant d\}$. (These subgroups are the groups denoted $P(\mathbb{Q})$ and $P(\mathbb{Z})$ in (4.1).) We fix r and let $U = G/C^{r+1}G$, $U(\mathbb{Q}) = G(\mathbb{Q})/C^{r+1}G(\mathbb{Q})$ and $U(\mathbb{Z}) = G(\mathbb{Z})/C^{r+1}G(\mathbb{Z})$. (By (4.5) and (2.7), $U(\mathbb{Q})$ and $U(\mathbb{Z})$ are subgroups of U.) Let E_r be as in (1.1) and let $A = \operatorname{End}_G(E_r)$. As a $\mathbb{C}$-algebra, $A = \mathbb{C}\langle D_1,\dots, D_d\rangle/J^{r+1}$ where D_p is as in (2.1) and J is the ideal generated by $\{D_p\}$. We let h_p be the image of g_p in U and E_p the image of D_p in A. Then $I = J/J^{r+1}$ is the ideal of A generated by $\{E_p\}$. By (2.5) we have an embedding $U \to 1 + I$. Then the complex Lie algebra $L \subseteq I$ generated by $\{E_p\}$ is isomorphic to $\operatorname{Lie}(U)$ via the embedding, and we let $L(\mathbb{Q})$ be the $\mathbb{Q}$-Lie algebra generated by $\{E_p\}$. (It follows from (4.5) that $L = L(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.) We let exp: $I \to 1 + I$ and log: $1 + I \to I$ be $\exp(1) = \sum (k!)^{-1} a^k$ and $\log(1 + a) = \sum (-1)^{k-1} k^{-1} a^k$, so $E_p = \log(h_p)$. Since $U \subseteq A$, (2.5), we have that $\exp \mid U = \exp_U$, and $\log = \exp_U^{-1}$ so $\exp(E_P) = h_p$. I has as basis the monomials in $\{E_p\}$ of degree at most r. Let $I(\mathbb{Q})$ be the $\mathbb{Q}$ span of this basis.

We also require the following terminology:

DEFINITION 5.2. A discrete group Γ is *uniquely divisible* if given $g \in \Gamma$ and $n \neq 0$ in $\mathbb{Z}$ there is a unique h in Γ with $h^n = g$.

We will require the following properties of uniquely divisible groups whose elementary proofs we omit:

LEMMA 5.3. *Let $1 \to K \to L \to M \to 1$ be an exact sequence of discrete groups.*

(1) *If K is central in L and K and M are uniquely divisible, then so is L.*

(2) *If L is uniquely divisible and M is torsion-free, then K is uniquely divisible.*

We use (5.3)(1) to see that $U(\mathbb{Q})$ is uniquely divisible. We need a remark about the lower central series of $U(\mathbb{Q})$, which we record in some generality for future reference.

PROPOSITION 5.4. *In Notation* 5.1 *for* $1 \leqslant s \leqslant r+1$,

(1) $C^sU(\mathbb{Q}) = (C^sU) \cap U(\mathbb{Q})$,

(2) $C^sU(\mathbb{Z}) = (C^sU) \cap U(\mathbb{Z})$,

(3) $C^sU(\mathbb{Q})/C^{s+1}U(\mathbb{Q}) = C^sG(\mathbb{Q})/C^{s+1}G(\mathbb{Q})$,

(4) $C^sU(\mathbb{Z})/C^{s+1}U(\mathbb{Z}) = C^sG(\mathbb{Z})/C^{s+1}G(\mathbb{Z})$.

Proof. (1) and (3) follow from (4.5) and (2) and (4) from (2.7).

COROLLARY 5.5. *$U(\mathbb{Q})$ is uniquely divisible.*

Proof. We prove that $C^sU(\mathbb{Q})$ is uniquely divisible by descending induction on s, the case $s = r+1$ being clear. We have an exact sequence $1 \to C^{s+1}U(\mathbb{Q}) \to C^sU(\mathbb{Q}) \to C^sU(\mathbb{Q})/C^{s+1}U(\mathbb{Q}) \to 1$ and then (5.4.3), (4.4), and (5.3)(1) give the result.

We also need to note how $U(\mathbb{Z})$ sits in $U(\mathbb{Q})$:

PROPOSITION 5.6. *In Notation* 5.1, *let $y \in U(\mathbb{Q})$. Then there is $n \neq 0$ in $\mathbb{Z}$ with $y^n \in U(\mathbb{Z})$.*

Proof. We use induction on r, and use the overbar to denote the homomorphism $U \to U/C^rU$. By induction, $\bar{y}^d = \bar{x}$ for some $x \in U(\mathbb{Z})$ and $d \neq 0$ in $\mathbb{Z}$, so $y^d = xu$ where $u \in C^rU(\mathbb{Q})$, which is central in $U(\mathbb{Q})$. It follows from (4.4) and (5.4) that the rational vector space $C^rU(\mathbb{Q})$ equals

$C^rU(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, so there is $s \neq 0$ in $\mathbb{Z}$ with $u^s \in C^rU(\mathbb{Z})$. Thus, $y^{ds} = u^s x^s \in U(\mathbb{Z})$.

We also need to detect the position of $L(\mathbb{Q})$ inside L.

LEMMA 5.7. *In Notation* 5.1, $\mathscr{C}^s L(\mathbb{Q}) = (\mathbb{C}^s L) \cap I(\mathbb{Q})$ *for* $1 \leqslant s \leqslant r+1$.

Proof. Since $L(\mathbb{Q})$ and L are truncations of free Lie algebras, it follows from [9, Example 2, L. 4.18] that it suffices to consider the case $s = 1$. Since $E_p \in L \cap I(\mathbb{Q})$, $L(\mathbb{Q}) \subseteq L \cap I(\mathbb{Q})$. In [9, Theorem 8.1, LA 4.15] there is defined a map $\phi: I \to L$ such that $\phi(I(\mathbb{Q})) \subseteq L(\mathbb{Q})$, and $\phi(X) = X$ if $X \in L$. It follows that $L \cap I(\mathbb{Q})$ is contained in $L(\mathbb{Q})$, so $L(\mathbb{Q}) = L \cap I(\mathbb{Q})$.

The next result is the key to our proof of the Malcev embedding:

PROPOSITION 5.8. *In Notation* 5.1, $\exp(L(\mathbb{Q})) = U(\mathbb{Q})$.

Proof. We note that $U(\mathbb{Q})$ and $\exp(L(\mathbb{Q}))$ are subsets of $1 + I(\mathbb{Q})$. In (3.17) we obtained a Lie algebra isomorphism Exp: $\mathrm{gr}(L) \to \mathrm{gr}(U)$ with $\mathrm{Exp}(X + \mathscr{C}^i L) = \exp(X)\, C^i G$ for $X \in \mathscr{C}^{i-1} L$. Now $\mathrm{gr}(L)$ is isomorphic to L and, hence, is generated by $\{E_p + \mathscr{C}^2 L\}$, and $\mathrm{Exp}(E_p + \mathscr{C}^2 L) = h_p C^2 U$. By (5.4.3) and (4.5) we have $\mathrm{gr}(U(\mathbb{Q})) \subseteq \mathrm{gr}(U)$ and, hence, $\mathrm{Exp}(\mathrm{gr}(L(\mathbb{Q}))) \subseteq \mathrm{gr}(U(\mathbb{Q}))$; in particular, $\exp(\mathscr{C}^r L(\mathbb{Q})) \subseteq C^r U(\mathbb{Q})$. Assume inductively that $\exp(\mathscr{C}^{k+1} L(\mathbb{Q})) \subseteq C^{k+1} U(\mathbb{Q})$ and let $X \in \mathscr{C}^k L(Q)$. Then, $\mathrm{Exp}(X + \mathscr{C}^{k+1} L) \in C^k U(\mathbb{Q})/C^{k+1} U(\mathbb{Q})$, so there is $y \in C^k U(\mathbb{Q})$ such that $z = \exp(X)\, y^{-1} \in C^{k+1} U$. Now $\exp(X)$ and y are both in $1 + I(\mathbb{Q})$, so z is in $(1 + I(\mathbb{Q})) \cap C^{k+1} U$ and, hence, $\log(z) \in I(\mathbb{Q}) \cap \mathscr{C}^{k+1} L$. By (5.7), we have $\log(z) \in \mathscr{C}^{k+1} L(\mathbb{Q})$, so by induction $z \in C^{k+1} U(\mathbb{Q})$, so $\exp(X) \in C^k U(\mathbb{Q})$. By induction we find $\exp(L(\mathbb{Q})) \subseteq U(\mathbb{Q})$. Now since $U(\mathbb{Q}) \subseteq 1 + I(\mathbb{Q})$, $\log(U(\mathbb{Q})) \in I(\mathbb{Q}) \cap L$ which is $L(\mathbb{Q})$ by (5.7), so $U(\mathbb{Q}) \subseteq \exp(L(\mathbb{Q}))$, establishing (5.8).

Because U and its subgroup $U(\mathbb{Q})$ are essentially free (with respect to being class r nilpotent), (5.8) extends directly to class r nilpotent uniquely divisible subgroups of unipotent groups.

PROPOSITION 5.9. *In Notation* 5.1, *let* U' *be a unipotent algebraic group, let* S *be a uniquely divisible subgroup of* U', *d-generated as a uniquely divisible group with* $C^{r+1} S = \{e\}$ *and let* $\bar{S}$ *be the Zariski-closure of* S *in* U'. *Then there exists a homomorphism* f: $U \to U'$ *with* $f(U) = \bar{S}$ *and* $f(U(\mathbb{Q})) = S$ *such that* $\exp^{-1}(S) = df(L(\mathbb{Q}))$ *is a rational Lie subalgebra of* $\mathrm{Lie}(U')$ *and* $\mathrm{Lie}(\bar{S}) = \mathbb{C} \exp^{-1}(S)$.

Proof. Suppose $y_1, \ldots, y_d$ generate S. Since $C^{r+1} S = \{e\}$, $C^{r+1} \bar{S} = \{e\}$ so there is f: $U \to \bar{S} \subseteq U'$ with $f(h_p) = y_p$. Then $f(U) = \bar{S}$. It is clear that $S' = f(U(\mathbb{Q}))$ is divisible, uniquely so since U' is. Since $y_p \in S'$, we have $S \subseteq S'$. It follows from (5.6) that S' is contained in any divisible subgroup of U'

containing $\{y_p\}$, so $S' \subseteq S$. Thus, $f(U(\mathbb{Q})) = S$. Since $\exp(L(\mathbb{Q})) = U(\mathbb{Q})$ by (5.8), $\exp^{-1}(S) = \exp^{-1}(f(U(\mathbb{Q})) = df(L(\mathbb{Q}))$ is a rational Lie algebra. Since $\mathbb{C}L(\mathbb{Q}) = L$, and $f(U) = \bar{S}$, we have $\mathrm{Lie}(\bar{S}) = \mathbb{C}\exp^{-1}(S)$.

It is also an immediate consequence of (5.8) and (5.9) that divisible subgroups of $U(\mathbb{Q})$ are finitely generated: For if $S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots$ is a proper ascending chain of finitely generated divisible subgroups of $U(\mathbb{Q})$, then $\exp^{-1}(S_0) \subsetneq \exp^{-1}(S_1) \subsetneq \cdots$ is a proper ascending chain of rational subalgebras of $L(\mathbb{Q})$ and, hence, stops.

With this observation and (5.9), we obtain an embedding theorem for uniquely divisible groups.

THEOREM 5.10. *Let H be a uniquely divisible nilpotent group, finitely generated as a uniquely divisible group. Then there is a unipotent algebraic group $\bar{H}$ and an injective homomorphism h: $H \to \bar{H}$ such that if g: $H \to U'$ is any homomorphhism from H to a unipotent group there is a unique homomorphism $\bar{g}$: $\bar{H} \to U'$ with $g = \bar{g}h$.*

Proof. Assume H is d-generated and $C^{r+1}H = \{e\}$. We can find a map $G(\mathbb{Q}) \to H$ whose image contains the generators of H, and whose kernel contains $C^{r+1}G(\mathbb{Q})$, so a map $U(\mathbb{Q}) \to H$ which by (5.5) must be onto. Let $R = \mathrm{Ker}(U(\mathbb{Q}) \to H)$; then by (5.3)(2), R is uniquely divisible and, as remarked above, finitely generated. Let $\bar{R}$ be the Zariski-closure of R. By (5.9) (with appropriate choice of d, r), $\mathrm{Lie}(\bar{R}) = \mathbb{C}\exp^{-1}(R)$. Now $\exp^{-1}(R) \subseteq L(\mathbb{Q})$ by (5.8), and $L = \mathbb{C} \otimes L(\mathbb{Q})$, so $\mathrm{Lie}(\bar{R}) = \mathbb{C} \otimes \exp^{-1}(R)$. Thus, Lie $(\bar{R}) \cap L(\mathbb{Q}) = \exp^{-1}(R)$, so $\bar{R} \cap U(\mathbb{Q}) = R$. Now $U(\mathbb{Q})$ is Zariski-dense in U and R is normal in $U(\mathbb{Q})$, so $\bar{R}$ is normal in U. Thus, $H = U(\mathbb{Q})/R$ embeds in the unipotent group $\bar{H} = U/\bar{R}$. It is clear that $\bar{H}$ has the desired universal property.

We abstract part of the argument of (5.10) for us below: we showed that if S is a divisible subgroup of $U(\mathbb{Q})$ with Zariski-closure $\bar{S}$, then $\bar{S} \cap U(\mathbb{Q}) = S$. We apply this remark to obtain the following information about Zariski-closures of subgroups of $U(\mathbb{Z})$:

LEMMA 5.11. *In Notation 5.1, let R be a normal subgroup of $U(\mathbb{Z})$ and $\bar{R}$ its Zariski-closure in U. Then $(\bar{R} \cap U(\mathbb{Z}))/R$ is finite.*

Proof. $U(\mathbb{Z})$ is finitely generated nilpotent, hence Noetherian, so R is finitely generated. Let S be the smallest uniquely divisible subgroup of $U(\mathbb{Q})$ containing R. Assume R (and hence, S) is generated by $y_1, \ldots, y_m$, and use prime (( )$'$) to indicate Notation (5.1) with d replaced by m. It follows from (5.9) that there is a homomorphism f: $U' \to U$ with $f(g_p) = y_p$, $f(U'(\mathbb{Z})) = R$ and $f(U'(\mathbb{Q})) = S$. If $x \in S$, it follows from (5.6) that there is $n \neq 0$ in $\mathbb{Z}$ with $x^n \in R$. Since the Zariski-closure of $S = f(U') = \bar{R}$, by the remark preceding the lemma we have $\bar{R} \cap U(\mathbb{Q}) = S$. Thus, $\bar{R} \cap U(\mathbb{Z})$ is contained in S. Thus, if $x \in \bar{R} \cap U(\mathbb{Z})$, there is $n \neq 0$ in $\mathbb{Z}$ with $x^n \in R$. Now $\bar{R} \cap U(\mathbb{Z})$ is also finitely generated, so $(\bar{R} \cap U(\mathbb{Z}))/R$ is finitely generated, nilpotent, and torsion, hence finite.

From (5.11) we deduce the embedding theorem for torsion free nilpotent groups:

THEOREM 5.12. *Let Γ be a finitely generated torsion free nilpotent discrete group. Then there is a unipotent algebraic group $\bar{\Gamma}$ and an injective homomorphism $h: \Gamma \to \bar{\Gamma}$ such that if $g: \Gamma \to U'$ is any homomorphism from Γ to a unipotent group there is a unique homomorphism $\bar{g}: \bar{\Gamma} \to U'$ with $g = \bar{g}h$.*

Proof. Assume Γ is d-generated and $C^{r+1}\Gamma = \{e\}$. We can find a surjection $U(\mathbb{Z}) \to \Gamma$ with kernel R. Let $\bar{R}$ be the Zariski-closure of R in U. Then $A = (\bar{R} \cap U(\mathbb{Z}))/R$ is finite by (5.11) and injects into $U(\mathbb{Z})/R = \Gamma$. Since Γ is torsion free, $A = \{e\}$ and $\bar{R} \cap U(\mathbb{Z})) = R$. Now R is normal in $U(\mathbb{Z})$ so $\bar{R}$ is normal in U, and $\Gamma = U(\mathbb{Z})/R$ injects into $\bar{\Gamma} = U/\bar{R}$. It is clear that $\bar{\Gamma}$ has the desired universal property.

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